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Confidence Bounds for Ratio of Variance Components in Unbalanced Two-Way Crossed Models

1. Introduction

The problem discussed here is that of obtaining confidence bounds for the proportion of total variances, p_A and p_B, for unbalanced two-way crossed classification components-of-variance models where some of the cells may be empty. These quantities are useful in many applications, specifically in genetics; see KEMPTHORNE (1952), OSBORNE and PATERSON (1952). The model, described in several text books (see SCHEFFE' ,1959; SEARLE, 1971; GRAYBILL, 1976) is

$$Y_{ijk} = \mu + A_i + B_j + E_{ijk}, \quad (i=1,2,\dots,a;\ j=1,2,\dots,b;\ k=1,\dots,n_{ij} \geq 0) \qquad (1.1)$$

where $A_1, A_2, \dots A_a; B_1, B_2, \dots, B_b; E_{111}, E_{112}, \dots, E_{11n_{11}}, E_{121}, \dots, E_{abn_{ab}}$ are unobservable random variables, jointly normally distributed, and pairwise uncorrelated with zero means. The variances of A_i, B_j and E_{ijk} are σ^2_A and σ^2_E respectively. The model in (1.1) can be written in matrix form as

$$\underline{Y} = \mu X_0 + X\left[\frac{\underline{\alpha}}{\underline{\beta}}\right] + \underline{E}$$

$X = [X_1, X_2]$; $\underline{Y} = [Y_{111}, Y_{112}, \dots, Y_{11n_{11}}, Y_{121}, \dots, Y_{12n_{12}}, \dots, Y_{abn_{ab}}]'$; μ is a constant; X_0 is an N x 1 vector with all elements 1; X_1 and X_2 are N x a and N x b known matrices respectively, with elements zeros or ones; $\underline{E}$ is an N x 1 vector with elements E_{ijk};

$N = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}$. Also

$$\underline{\alpha} \sim MVN(\underline{0}, \sigma^2_A I_a);\ \underline{\beta} \sim MVN(\underline{0}, \sigma^2_B I_b);\ \underline{E} \sim MVN(\underline{0}, \sigma^2_E I_N);$$

$\underline{\alpha}$, $\underline{\beta}$ and $\underline{E}$ are independent. $E(\underline{Y}) = \mu X_0$ and
$var(\underline{Y}) = \Sigma = X_1X'_1\sigma^2_A + X_2X'_2\sigma^2_B + I_N\sigma^2_E$.

Since no exact confidence bounds are available on $p_A = \sigma^2_A/\sigma^2$ and $p_B = \sigma^2_B/\sigma^2$ for this model we obtain approximate confidence bounds. Approximate methods have been developed to construct confidence bounds on p_A and p_B in the balanced complete two-way crossed models, $n_{ij} = m$ for all i and j (see ARTEAGA, JEYARATNAM and GRAYBILL, 1982).

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SRINIVASAN and GRAYBILL (1986) obtained approximate confidence bounds on p_A and p_B for the model in (1.1) when $n_{ij} > 0$ for all i and j.

2. Derivation of the Confidence Bounds

In cases where $n_{ij} = 1$ for all i and j, **ARTEAGA**, **JEYARATNAM** and **GRAYBILL** (**AJG**, 1982) derived good approximate upper, lower and two-sided confidence bounds for p_A and p_B. For the case they considered the Analysis of Variance (ANOVA) is given in Table 1 below.

Table 1
ANOVA for Balanced Crossed Components-of-Variance Model

Source of variation	Degrees of freedom	Mean Square (MS)	E(MS)
Factor A	$n_1 = a - 1$	$S_1^2 = \frac{\Sigma\Sigma(\bar{Y}_{i.} - \bar{Y}_{..})^2}{n_1}$	$\Theta_1 = \sigma_E^2 + b\sigma_A^2$
Factor B	$n_2 = b - 1$	$S_2^2 = \frac{\Sigma\Sigma(\bar{Y}_{.j} - \bar{Y}_{..})^2}{n_2}$	$\Theta_2 = \sigma_E^2 + a\sigma_B^2$
Error	$n_3 = n_1 \cdot n_2$	$S_3^2 = \frac{\Sigma\Sigma(\bar{Y}_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2}{n_3}$	$\Theta_3 = \sigma_E^2$

For the balanced model, sums of squares are uniquely defined and possess several useful properties, such as $n_i S_i^2/\Theta_i$, $i=1,2$ are distributed as independent chi-squares with n_i degrees of freedom. The $1 - \alpha$ approximate lower confidence bound for p_A in AJG (1982) is $L_A(R_1^2, R_2^2)$ where

$$L_A(R_1^2, R_2^2) = \frac{aB(R_1^2, R_2^2)}{b + aB(R_1^2, R_2^2)}, \tag{2.1}$$

$R_i^2 = \frac{S_i^2}{S_3^2}$, $i = 1,2$ and

$$B(R_1^2, R_2^2) = \frac{-1 + F^{-1}_{a:n_1,\infty} R_2^2 + (1 - F_{a:n_1,\infty} F_{a:n_1,n_3})\, F_{a:n_1,n_3} R_1^{-2}}{n_1 + F_{a:n_1,\infty} F_{a:n_1,n_2} R_2^2}.$$

A $1 - \alpha$ approximate upper confidence bound for p_A is obtained by replacing α with $1 - \alpha$ throughout in (2.1). Using numerical integration AJG showed that these confidence bounds are quite good.

For the unbalanced case; $n_iS_i^2/\Theta_i$, i = 1,2 may not be distributed as chi-square; also S_1^2 and S_2^2 may not be independent. In this paper we define sums of squares $n_AS_A^2$,$n_BS_B^2$ and $n_ES_E^2$ for the unbalanced model (1.1) and substitute them for $n_1S_1^2$, $n_2S_2^2$ and $n_3S_3^2$ respectively into (2.1) to obtain confidence bounds for p_A.
In unbalanced models there are several different mean squares that could be used to replace the S_i^2, i=1, 2, 3, but the ones used in this paper are obtained by considering the A_i and B_j in (1.1) to be fixed parameters (i. e. EISENHART model I). We get two ANOVA's depending on whether one wants to test the A_i's equal or B_j's equal (KEMPTHORNE, 1952). They are given in Table 2 below.

Table 2

ANOVA for testing A_i's equal			ANOVA for testing B_j's equal		
Sources	DF	SS	Source	DF	SS
Total	N - 1	$\sum_{i,j,k,} (Y_{ijk} - \overline{Y}_{...})^2$	Total	N - 1	$\sum_{i,j,k} (Y_{ijk} - \overline{Y}_{...})^2$
A (adjusted for B)	a - 1	$\underline{Y}'A\underline{Y}$	A (ignoring B)	b - 1	$\sum_{i,j,k} (\overline{Y}_{i..} - \overline{Y}_{...})^2$
B (ignoring A)	b - 1	$\sum_{i,j,k} (\overline{Y}_{.k.} - \overline{Y}_{...})^2$	B (adjusted for A)	a -1	$\underline{Y}'B\underline{Y}$
Error	$n_{E(A)}$	$n_{E(A)}S^2_{E(A)}$	Error	$n_{E(B)}$	$n_{E(B)}S^2_{E(B)}$

Errors terms are obtained by subtraction. The matrices A and B and the means of squares S_A^2, S_B^2 are defined by

$$A = XX^- - X_2X_2^-, \qquad \underline{Y}'\frac{A}{a-1}\underline{Y} = S_A^2$$

$$B = XX^- - X_1X_1^-, \qquad \underline{Y}'\frac{B}{b-1}\underline{Y} = S_B^2$$

where G^- denotes the Moore-Penrose inverse of the matrix G (GRAYBILL, 1976). It is well known that

i) $n_{E(A)} = n_{E(B)} = N - a - b + 1$ which we denote by n_E

ii) $S^2_{E(A)} = S^2_{E(B)} = \underline{Y}'\frac{(I-XX^-)}{N-a-b+1}Y$ which we denote by S_E^2.

The quantities S_A^2, S_B^2 and S_E^2 defined above are organized into an ANOVA given in Table 3.

Table 3

Sources of Variation	DF	MS	E(MS)
Total	$\sum_{i,j} n_{ij} = N$		
Mean	1		
Factor A (adjusted for B)	$n_A = a - 1$	$S_A^2 = \underline{Y}'\,A\underline{Y}/n_A$	$\Theta_A = \sigma_E^2 + d_1\sigma_A^2$
Factor B (adjusted for A)	$n_B = b - 1$	$S_B^2 = \underline{Y}'\,B\underline{Y}/n_B$	$\Theta_B = \sigma_E^2 + d_2\sigma_B^2$
Error	$n_E = N - a - b + 1$	$S_E^2 = \underline{Y}'\frac{I-XX^-}{n_E}\underline{Y}$	$\Theta_E = \sigma_E^2$

where $d_1 = \text{trace}\left(\frac{X'_1(I-X_2X_2^-)X_1}{a-1}\right)$, $d_2 = \text{trace}\left(\frac{X'_2(I-X_1X'_1)X_2}{b-1}\right)$. Notice that the sums of squares for mean, factor A, factor B and error may not sum to the total in Table 3; they do if $n_{ij} = 1$ for all i and j (i. e. in this case $S_A^2 = S_1^2$, and $S_B^2 = S_2^2$ in Table 1 for the balanced case).

The expected mean square columns in Tables 2 and 3 are obtained by assuming the variance component model in (1.1). The following results are proved in KAZEMPOUR (1987) and are used in obtaining the confidence bounds.

(i) S_A^2, S_B^2 and S_E^2 may not be independent nor scaled chi-square random variables.

(ii) S_A^2 and S_B^2 are independent of S_E^2.

(iii) S_A^2 and S_B^2 are independent if $\Theta_E = 0$.

In the confidence bounds exhibited in (2.1), S_1^2 and S_2^2 are scaled chi-square random variables. Since S_A^2 and S_B^2 which will replace S_1^2 and S_2^2, may not be scaled chi-square random variables for the general unbalanced model we use a Satterthwaite type procedure (SATTERTHWAITE, 1946) to find an approximate distribution for S_A^2 and S_B^2. To do this we set the first two moments of $n_A S_A^2/E(S_A^2)$ equal to the first two moments of a chi-square random variable with n_A degrees of freedom and solve for n_A. We call $n_A S_A^2/E(S_A^2)$ an "approximate chi-square" random variable. Note that $n_E S_E^2/E(S_E^2)$ is distributed as a chi-square random variable with n_E degrees of freedom and no approximation is needed.

By straight forward but tedious algebraic it can be shown (KAZEMPOUR, 1987) that if $n_A S_A^2/E(S_A^2)$ is an "approximate chi-square" random variable then

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$$n_A = \frac{2[E(S^2{}_A)]^2}{var(S^2{}_A)} = \frac{a - 1}{1 + \left[\frac{(a-1)tr\,(T^2{}_1)}{(tr(T_1))^2} - 1\right]\ [1 / 1 + \gamma\,(a - 1)/tr\,(T_1)]} \tag{2.2}$$

where $\gamma = \sigma^2_A/\sigma^2_E$, $T_1 = X'_1\,(XX^- - X_2X_2^-)\,X_1$. n_A can serve as degrees of freedom for the approximation, but it depends on the unknown parameters σ^2_A and σ^2_E. If σ^2_A and σ^2_E are replaced with their estimators, then an estimator for n_A, say $\hat{n}_A$, can be obtained. To find the range of n_A as σ^2_A/σ^2_E varies from 0 to ∞ we note that n_A is a monotone function of $\gamma = \sigma^2_A/\sigma^2_E$; also note that for any symmetric matrix P, the following relation holds (GRAYBILL, 1983).

$$[tr\,(P))]^2 \leq rank\,(P) \cdot tr\,(P^2).$$

Therefore

$$\frac{(a-1)\,tr\,(T^2{}_1)}{(tr\,(T_1))^2} \geq 1,$$

and for $\gamma = 0$, n_A reaches its minimum,

$$(a - 1) / [\,(a - 1)\,tr\,(T^2{}_1) / (tr\,(T_1))^2].$$

If $\gamma \to \infty$, n_A reaches its maximum, a - 1. Therefore

$$(tr\,(T_1))^2 / tr\,(T^2{}_1) \leq n_A \leq a - 1.$$

It is clear that for complete balanced models, $n_{ij} = 1$ for all i and j, $n_A = a - 1$. Also it can be proved (KAZEMPOUR, 1987) that if the design is a Balanced Incomplete Block (BIB), with equal number of blocks and treatments, then $n_A = a - 1$.

By a similar procedure $n_B S^2_B/E(S^2_B)$ can be viewed as an "approximate chi-square" random variable with n_B degrees of freedom, also it can be shown that $(tr\,(T_2))^2 / tr\,(T^2_2) \leq n_B \leq b - 1$. If S^2_A, S^2_B, S^2_E, n_A and n_B are substituted for S^2_1, S^2_2, S^2_3, n_1 and n_2 respectively into $L_A\,(R^2_1, R^2_2)$ in (2.1) we obtain $L^*_A(R^2_A, R^2_B)$, an approximate $1 - \alpha$ lower confidence bound on p_A for the model in (1.1).

$$L^*_A(R^2_A,R^2_B) = \frac{d_2B^*(R^2_A,R^2_B)}{d_1 + d_2B^*(R^2_A,R^2_B)} \tag{2.3}$$

where

$$B^*(R^2_A,R^2_B) = \frac{-1 + F_{a:n_A,\infty}\,R^2_A + (1 - F_{a:n_A,\infty}\,F_{a:n_A,n_E})\,F_{a:n_A,n_E}R^{-2}_A}{(d_2 - 1) + F_{a:n_A,\infty}F_{an_A,n_B}R^2_B},$$

and

$$R^2_A = \frac{S^2_A}{S^2_E}, \qquad R^2_B = \frac{S^2_B}{S^2_E}.$$

To compute the bounds one could use the maximum values of n_A and n_B, or the minimum values of n_A and n_B or an estimate of n_A and n_B by using estimators for σ^2_A, σ^2_B and σ^2_E in (2.2). To obtain a 1- α approximate upper confidence bound, U^*_A, replace a by 1 - α in the above equation. Also 1 - 2α approximate two sided confidence bounds for p_A is (L^*_A, U^*_A). For confidence bounds on p_B, (2.3) is used by interchanging S^2_A and S^2_B, and interchanging n_A and n_B.

3. Simulation and Evaluation

To evaluate the confidence bounds in (2.3) a simulation study was conducted. The distribution of the bounds and the confidence coefficients depend on unknown parameters, σ^2_A, σ^2_B and σ^2_E through the ratio p_A and p_B. To examine the effects of the dependency we let p_A and p_B assume a subset of their allowable values. Since $0 \leq p_A \leq 1$; $0 \leq p_B \leq 1$ and $p_A + p_B \leq 1$ the values assigned to them are from the set {.05, .2, .4, .6, .8, .9} with the restriction $p_A + p_B < 1$. So the values assigned to (p_A, p_B) as pairs are the following 17 pairs
{(.05,.05), (.05,.2), (.05,.4), (.05,.6), (.05,.8), (.05,.9), (.2,.05), (.2,.2), (.2,.4), (.2,.6), (.4,.05), (.4,.2), (.4,.4), (.6,.05), (.6,.2), (.8,.05), (.9,.05)}.
Several sets of values for a, b and n_{ij}, i = 1, 2, .. a; j = 1,2, ... b were used. Each of these sets can be viewed as a configuration for a Randomized Block Design (MONTGOMERY, 1984) with b blocks, a treatments and n_{ij} elements per cell, (see appendix).

For each pair of (p_A, p_B) and each set of the configuration a random vector, $\underline{Y}$, of size

$$N = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}$$ was generated from normal distribution with mean zero and variance-covariance $X_1X_1'p_A + X_2X_2'p_B + I(1 - p_A - p_B)$. IMSL was used to generate random vectors.

For each chosen pair of (p_A, p_B) and each configuration of a, b, n_{ij}, i = 1, ..., a; j = 1, 2, ... , b, one thousand $\underline{Y}$ values were generated S^2_A, S^2_B, S^2_E were calculated and used in (2.3) to obtain the confidence bounds. Each bound was checked to see if it covered the true values of p_A. The number of times that the proposed bound covers the true value divided by 1000 is an approximation for the nominal confidence coefficient. This procedure was repeated seventeen times, the number of pairs of (p_A, p_B), for each design configuration and seventeen confidence coefficients were obtained. The smallest and the largest of these observed confidence coefficients are reported in Table 5.1 - 5.6 for each configuration.
To evaluate the proposed confidence bounds twenty three different design configurations were used. Nine Balanced Randomized Block (BRB) designs (Table 5.1 - 5.2), six Balanced Incomplete Block (BIB) designs (Table 5.3 - 5.4) and eight Unbalanced Randomized Block designs (Table 5.5 - 5.6) were examined in the simulation study.
The first two columns of Tables 5.1 - 5.4 list the number of blocks and the number of treatments. In BRB design n_{ij} = 1 for all i and j, but n_{ij} = 0 and/or 1 for all i and j in BIB designs.

Columns 3 and 4 in Table 5.1 show the smallest and the largest confidence coefficients obtained from 17 different values of (p_A, p_B) for lower and upper bounds when $\alpha = .05$. The third column in Table 5.2 shows the smallest and the largest observed confidence coefficients for two-sided bounds and $\alpha = .10$. An extra column is introduced in Table 5.3 and 5.4 to indicate λ, the number of times that each pair of treatment appears in the same block [5]. Entries in column 4 and 5 in Table 5.3 are the ranges observed confidence coefficients of the lower and upper bounds for p_A and $\alpha = .05$. The same is true for the 4th column of Table 5.4 when $\alpha = .10$ in two-sided bounds. For some general unbalanced two-way designs the results are in Table 5.5 - 5.6. The first column of tables 5.5 - 5.6 list the design numbers which refer to incidence matrices in the appendix. Columns 2, 3, and 4 present the number of blocks, treatments and observations respectively. The fifth and sixth columns show the ranges of the confidence coefficients for (as p_A and p_B vary over their allowable values) lower and upper bounds $\alpha = .05$. For the same design configurations Table 5.5 the range of two-sided bounds are given in the last column of Table 5.6 when $\alpha = .1$. From Tables 5.1 - 5.6 it appears that the lower, upper and two-sided confidence bounds obtained by (2.3) result in confidence coefficients close enough to the nominal confidence coefficients for all 23 different configurations to be useful. For designs not studied in this paper, simulation results can be obtained to determine if the bounds in (2.3) give confidence coefficients close enough to the nominal values to be useful.

Table 5.1
Range of Coefficients for 95% One-Sided Confidence Bounds for p_A for Two-Way Balanced Complete Models

Number of Blocks	Number of Treatments	Lower Bound	Upper Bound
3	3	.943-.953	.949-.983
3	7	.938-.963	.946-.963
3	11	.936-.957	.944-.963
5	3	.939-.963	.941-.983
5	11	.951-.961	.951-.961
7	3	.941-.952	.948-.983
7	11	.941-.959	.944-.907
9	3	.938-.967	.952-.990
9	11	.941-.957	.944-.971

Table 5.2
Range of Coefficients for 90% Two-Sided Confidence Bounds for p_A for Two-Way Balanced Complete Models

Number of Blocks	Number of Treatments	Range of Coefficients
3	3	.901-.932
3	7	.900-.913
3	11	.899-.908
5	3	.900-.938
5	11	.899-.913
7	3	.900-.943
7	11	.898-.916
9	3	.900-.946
9	11	.898-.921

Table 5.3
Range of Coefficients for 95% One-Sided Confidence Bounds for p_A for Two-Way Balanced Incomplete Block

Number of Blocks	Number of Treatments	λ*	Lower Bound	Upper Bound
6	4	1	.945-.985	.937-.951
7	7	2	.949-.978	.944-.961
10	6	2	.940-.969	.951-.969
11	11	2	.952-.972	.944-.960
15	6	6	.935-.968	.938-.977
20	6	4	.948-.986	.949-.964

*λ is the number of times that each pair of treatment appears in the same block.

Table 5.4
Range of Coefficients for 90% Two-Sided Confidence Bounds for p_A for Two-Way Balanced Incomplete Block

Number of Blocks	Number of Treatments	λ	Two-Sided
6	4	1	.894-.931
7	7	2	.894-.936
10	6	2	.897-.934
11	11	2	.903-.923
15	6	6	.886-.939
20	6	4	.897-.948

Table 5.5
Range of Coefficients for 95% One-Sided Confidence Bounds for p_A forTwo-Way Unbalanced Models

Design Number*	Number of Blocks	Number of Treatments	Number of Observations	Lower Bound	Upper Bound
D1	3	3	40	.931-.962	.948-.961
D2	3	8	55	.939-.956	.942-.964
D3	5	5	80	.944-.960	.946-.966
D4	5	3	44	.937-.958	.944-.960
D5	8	3	49	.936-.962	.949-.974
D6	3	5	35	.940-.957	.943-.961
D7	4	5	18	.957-.983	.944-.963
D8	3	8	79	.953-.973	.937-.962

* These designs are displayed in the appendix.

Table 5.6
Range of Coefficients for 90% Two-Sided Confidence Bounds for p_A for Two-Way Unbalanced Model

Design Number	Number of Blocks	Number of Treatments	Number of Observations	Two-Sided
D1	3	3	40	.899-.936
D2	3	8	55	.879-.930
D3	5	5	80	.900-.935
D4	5	3	44	.894-.922
D5	8	3	49	.893-.962
D6	3	5	35	.894-.922
D7	4	5	18	.915-.945
D8	3	8	79	.894-.930

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Summary

Consider the two-way crossed components-of-variance model given by

$$y_{ijk} = \mu + A_i + B_j + E_{ijk} \; (i=1,2, \ldots , a; j=1,2, \ldots, b; k=1,2, \ldots ,n_{ij} \geq 0)$$

where A_i, B_j and E_{ijk} are unobservable independent normal random variables with means zero and variances σ^2_A, σ^2_B and σ^2_E respectively. In this paper confidence bounds on $p_A = \sigma^2_A/\sigma^2$ and $p_B = \sigma^2_B/\sigma^2$, where $\sigma^2 = \sigma^2_A + \sigma^2_B + \sigma^2_E$, are derived for the above model. The balanced incomplete block model is a special case.

Keywords : Confidence intervals for variance components, unbalanced two-way models, empty cells.

Zusammenfassung

Titel der Arbeit: Klassifikationsmodell für Varianzkomponenten (Modell II für eine 2-fache Kreuzklassifikation)

Betrachtet wird das durch $y_{ijk}=\mu+A_i+B_j+E_{ijk}$ $(i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,\ n_{ij} \geq 0)$ gegebene Modell II für eine 2-fache Kreuzklassifikation von Varianzkomponenten wobei A_i, B_j und E_{ijk} nichtbeobachtbare unabhängige normalverteilte Zufallsvariable mit den Mittelwerten Null und den Varianzen σ^2_A, σ^2_B bzw. σ^2_E sind. In diesem Beitrag werden für das Modell Konfidenzgrenzen an den Stellen $p_A = \sigma^2_A/\sigma^2$ und $p_B = \sigma^2_B/\sigma^2$ mit $\sigma^2 = \sigma^2_A + \sigma^2_B + \sigma^2_E$ abgeleitet. Die balancierte unvollständige Blockanlage stellt einen besonderen Fall dar.

Schlüsselworte: Konfidenzintervalle für Varianzkomponenten, 2-fache Kreuzklassifikation, balancierte unvollständige Blockanlage

References

ARTEAGA, C.; JEYARATNAM, S.; GRAYBILL, F. A.: Confidence Intervals for Proportions of Total Variance in the Two-Way Cross Components of Variance Model. Communications in Statistics: Theory and Methods II, (1982), 1643 - 1658

GRAYBILL, F. A.: Theory and Application of the Linear Model. Duxbury Press, North Scituate, Massachusetts, 1976

KAZEMPOUR, M. K.: Confidence Bounds on Functions of Variance Components in Unbalanced Models. Ph.D. Thesis, Colorado State University, 1987

KEMPTHORNE, O.: The Design and Analysis of Experiments. John Wiley: New York, 1952

MONTGOMERY, D. C.: Design and Analysis of Experiments. John Wiley: New York, 1984

OSBORNE, R.; PATERSON, W. S. B.: On the Sampling Variance of Heritability Estimates Derived from Variance Analysis. Proceedings of the Royal Society. Edinburgh B LXIV (1952), 456 - 461

SATTERTHWAITE, F. E.: An Approximate Distribution of Estimates of Variance Components, Biometrics Bulletin (1946) 2, 110 - 114

SCHEFFE', H.: The Analysis of Variance. John Wiley: New York, 1970

SEARLE, S. R.: Linear Models. John Wiley: New York, 1970

SRINIVASAN, S.: Confidence Intervals on Functions of Variance Components in Unbalanced Two-Way Design Models. Ph.D. Thesis, Colorado State University, 1986

Appendix

Incidence matrices for different designs which were used are presented in this appendix. Rows are blocks and columns are treatments. The entry is the number of observations in the cell.

Design # 1

$$D_1 = \begin{bmatrix} 6 & 2 & 2 \\ 5 & 4 & 8 \\ 3 & 8 & 2 \end{bmatrix}$$

Design # 2

$$D_2 = \begin{bmatrix} 5 & 1 & 3 & 4 & 7 & 4 & 2 & 9 \\ 6 & 1 & 3 & 7 & 1 & 10 & 1 & 7 \\ 1 & 5 & 1 & 4 & 1 & 1 & 3 & 2 \end{bmatrix}$$

Design # 3

$$D_3 = \begin{bmatrix} 4 & 3 & 1 & 5 & 2 \\ 6 & 3 & 6 & 4 & 3 \\ 5 & 2 & 1 & 3 & 4 \\ 5 & 2 & 2 & 1 & 3 \\ 4 & 1 & 3 & 2 & 5 \end{bmatrix}$$

Design # 4

$$D_4 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 5 \\ 1 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 3 \end{bmatrix}$$

Design # 5

$$D_5 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \\ 2 & 1 & 4 \\ 1 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

Design # 6

$$D_6 = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 2 & 4 & 2 & 1 & 3 \\ 5 & 2 & 4 & 3 & 1 \end{bmatrix}$$

Design # 7

$$D_7 = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 2 \end{bmatrix}$$

Design # 8

$$D_8 = \begin{bmatrix} 0 & 0 & 2 & 3 & 0 & 2 & 40 & 3 \\ 3 & 2 & 0 & 0 & 2 & 4 & 5 & 2 \\ 4 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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